Bounds on Torsion Groups From Geometric Isogeny Classes

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(50-minute version)
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- Given a number field F, an **elliptic curve defined over** F is an algebraic curve $E_{/F}$ defined by

$$y^2 = x^3 + Ax + B$$

for some $A, B \in F$ where $4A^3 - 27B^2 \neq 0$.

  - Unless otherwise stated, all elliptic curves in this talk are defined over number fields!

- Unlike other algebraic curves, elliptic curves (genus $= 1$) are endowed with a **geometric group law.**

For an elliptic curve $E_{/F}$, we will use $E(F)$ to denote the group of F-rational points on E.

- $P := (x, y) \in E$ is F-**rational** if $x, y \in F$.

Theorem (Mordell-Weil).

Let F be a number field. Then for all elliptic curves $E_{/F}$, the group $E(F)$ is finitely generated.

- The **torsion subgroup of** E is the subgroup of points with finite order.

Corollary.

With notation as above, the torsion subgroup $E(F)$[tors] is finite.

Fixing the degree:

Elliptic curves over number fields satisfy a "strong uniform boundedness" in their torsion subgroups.

Theorem (Merel, 1996).

For each $d \in \mathbb{Z}^+$, there exists a constant $B := B(d) \in \mathbb{Z}^+$ so that for all elliptic curves $E_{/F}$ where $[F : \mathbb{Q}] = d$ one has

$$\#E(F)[\text{tors}] \leq B := B(d).$$

- "Fix the number field degree, and get a uniform bound."
- Examples: for $E_{/F}$, one has:
  - $[F : \mathbb{Q}] = 1 \Rightarrow \#E(F)[\text{tors}] \leq 16$;
  - $[F : \mathbb{Q}] = 2 \Rightarrow \#E(F)[\text{tors}] \leq 24$;
  - $[F : \mathbb{Q}] = 3 \Rightarrow \#E(F)[\text{tors}] \leq 28$.

Fixing the elliptic curve:

Not fixing the number field degree, but instead fixing $E_{/F}$ and studying $E(L)[\mathrm{tors}]$ where L varies:

- For each $M \in \mathbb{Z}^+$ there exists torsion points $R \in E$ with $[F(R) : F] > M$.

Do we have any control over $\#E(L)[\mathrm{tors}]$ when varying L?

Torsion under isogenies

Let's make this question harder!

- For the rest of this talk, fix an algebraic closure $\overline{\mathbb{Q}}$.
- Recall that an *isogeny* between two elliptic curves E_1, E_2 is a morphism $\phi : E_1 \to E_2$ which fixes basepoints.
  - ϕ is also a group homomorphism.
- The "$\overline{\mathbb{Q}}$-isogeny class of E" is the collection of elliptic curves E' which are $\overline{\mathbb{Q}}$-isogenous to E.
- As an adjective, *geometric* means $\overline{\mathbb{Q}}$-rational.

- Any $\overline{\mathbb{Q}}$-isogeny class of elliptic curves has j-invariants of arbitrarily large degree over $\mathbb{Q}$.
  - Non-CM elliptic curves: use Serre's open image theorem.
  - CM elliptic curves: use that class numbers of imaginary quadratic order tend to infinity as their discriminants $\Delta \to -\infty$.

  Do we have any control over $\#E(L)[\mathrm{tors}]$ when varying L and E in a fixed $\overline{\mathbb{Q}}$-isogeny class?

Theorem 1 (G., 2022).

Polynomial bounds: *Fix a $\overline{\mathbb{Q}}$-isogeny class $\mathcal{E}$. Then for each $\epsilon > 0$ there exists a constant $C_\epsilon := C_\epsilon(\mathcal{E})$ such that for all elliptic curves $E_{/F} \in \mathcal{E}$ one has*

$$\#E(F)[\text{tors}] \leq C_\epsilon \cdot [F : \mathbb{Q}]^{2+\epsilon}.$$

Theorem 2 (G., 2022).

Typical boundedness: *Fix a number field F_0. Then for each $\epsilon > 0$ there exists a constant $B_\epsilon := B_\epsilon(F_0)$ such that for all elliptic curves $E_{/F}$ $\overline{\mathbb{Q}}$-isogenous to some elliptic curve E' with $j(E') \in F_0$, one has*

$$\#E(F)[\text{tors}] \leq B_\epsilon$$

when $[F : \mathbb{Q}]$ does not lie in a certain subset of $\mathbb{Z}^+$ of upper density $\leq \epsilon$.

Polynomial Bounds on Torsion From Fixed Isogeny Classes

Recall:

Theorem (Merel, 1996).

For each $d \in \mathbb{Z}^+$, there exists a constant $B := B(d) \in \mathbb{Z}^+$ so that for all elliptic curves $E_{/F}$ where $[F : \mathbb{Q}] = d$ one has

$$\#E(F)[\text{tors}] \leq B := B(d).$$

- (Parent, 1999) If $p^n \mid \#E(F)[\text{tors}]$ then

$$p^n \leq 129(5^d - 1)(3d)^6$$

where $d := [F : \mathbb{Q}]$.

Folklore Conjecture (Clark, Cook and Stankewicz, 2013).

There exists a constant $\alpha > 0$ such that for all elliptic curves $E_{/F}$, one has

$$\#E(F)[\text{tors}] \leq [F : \mathbb{Q}]^{\alpha}.$$

In support of this conjecture: for an elliptic curve $E_{/F}$ and $d := [F : \mathbb{Q}] > 1$,

1. (Hindry and Silverman, 1999) if $j(E)$ is integral then

$$\#E(F)[\text{tors}] \leq 1977408 \cdot d \log d;$$

2. (Clark and Pollack, 2015) if E has CM then

$$\#E(F)[\text{tors}] \leq C \cdot d \log \log d$$

for some effectively computable constant $C \in \mathbb{Z}^+$.

- For a finite abelian group G, we let $\exp G$ denote the *exponent* of G.

3. (Clark and Pollack, 2018) For each $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for all elliptic curves $E_{/F}$ *base-changed from* $\mathbb{Q}$ one has

$$\exp E(F)[\mathrm{tors}] \leq C_\epsilon \cdot d^{3/2+\epsilon}$$

and

$$\#E(F)[\mathrm{tors}] \leq C_\epsilon \cdot d^{5/2+\epsilon}$$

where $d := [F : \mathbb{Q}]$.

More generally:

4. (Clark and Pollack, 2018) Fix a number field F_0. Assume that A. GRH is true, and that B. F_0 contains no Hilbert class fields of imaginary quadratic fields. Then for each $\epsilon > 0$ there exists $C_\epsilon > 0$ such that for all elliptic curves $E_{/F}$ *with* F_0*-rational* j*-invariant*, one has both

$$\exp E(F)[\mathrm{tors}] \leq C_\epsilon \cdot d^{3/2+\epsilon}$$

and

$$\#E(F)[\mathrm{tors}] \leq C_\epsilon \cdot d^{5/2+\epsilon}$$

where $d := [F : \mathbb{Q}]$.

- A. and B. are the **LV Hypotheses.**

To add to these results:

Theorem 1 (G., 2022).

Fix a number field F_0 and an elliptic curve E_{0/F_0}. Then for each $\epsilon > 0$ there exists $C_\epsilon := C_\epsilon(E_0, F_0) > 0$ such that for all elliptic curves $E_{/F}$ geometrically isogenous to E_{0/F_0} one has both

$$\exp E(F)[\text{tors}] \leq C_\epsilon \cdot d^{1+\epsilon}$$

and

$$\#E(F)[\text{tors}] \leq C_\epsilon \cdot d^{2+\epsilon}$$

where $d := [F : \mathbb{Q}]$.

Proof sketch

1. For each $\epsilon > 0$,

$$\exp E(F)[\mathrm{tors}] \leq d^B \Rightarrow \#E(F)[\mathrm{tors}] \leq C_\epsilon \cdot d^{B+1+\epsilon}$$

for some $C_\epsilon > 0$.

2. Assume E is non-CM. We then take for a certain $M \in \mathbb{Z}^+$

$$\exp E(F)[\mathrm{tors}] = \exp E(F)[M^\infty] \cdot \exp E(F)[M'].$$

Here, $E(F)[M^\infty]$ is the "M-primary torsion" and $E(F)[M']$ is the torsion supported away from M.

- M depends on E_{0/F_0} (it is the *adelic level of* E_{0/F_0}).

3. We will bound $\exp E(F)[M^\infty]$ and $\exp E(F)[M']$ independently!

Uniformly bounded support

Theorem 3 (G., 2022; variant of Clark and Pollack, 2018).

For integers $d_0, M > 0$, there exists a constant $C := C(d_0, M) > 0$ such that for any non-CM elliptic curve $E_{/F}$ geometrically isogenous to an elliptic curve whose j-invariant j_0 has degree $[\mathbb{Q}(j_0) : \mathbb{Q}] = d_0$, one has

$$\exp E(F)[M^\infty] \leq C \cdot \sqrt{d}$$

where $d := [F : \mathbb{Q}]$.

N-adic representations under isogeny

- To bound $\exp E(F)[M']$, we will exploit a relation between N-adic representations of E and E_0.
- Given an elliptic curve $E_{/F}$, for each integer $N \in \mathbb{Z}^+$ the absolute Galois group $G_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ acts on the *N-adic Tate module*

$$T_N(E) := \varprojlim_{k \geq 1} E[N^k],$$

giving rise to the N-adic representation

$$\rho_{E,N^\infty} : G_F \to \mathrm{GL}_2(\mathbb{Z}_N).$$

Theorem 4 (G., 2022, à la Greenberg, 2012).

Let $E_{/F}$ and $E'_{/F}$ be F-rationally isogenous non-CM elliptic curves. Then for all integers $N \in \mathbb{Z}^+$ one has

1. $[\mathrm{GL}_2(\mathbb{Z}_N) : \rho_{E,N^\infty}(G_F)] = [\mathrm{GL}_2(\mathbb{Z}_N) : \rho_{E',N^\infty}(G_F)]$;
2. $[\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) : \rho_{E,N}(G_F)] \mid [\mathrm{GL}_2(\mathbb{Z}_N) : \rho_{E',N^\infty}(G_F)]$.

- Can use this to show that for each $\epsilon > 0$, one has

$$\exp E(F)[M'] \leq c_\epsilon(d_0) \cdot d^{1/2+\epsilon}$$

  for some constant $c_\epsilon(d_0) > 0$.
- Combine this with $\exp E(F)[M^\infty] \leq C \cdot \sqrt{d}$.

- Conclusion: we have polynomial bounds on each geometric isogeny class of elliptic curves over number fields.
- Next, we'll describe uniformity results on torsion from all "F_0-rational" geometric isogeny classes.

Typically Bounding Torsion From $\mathcal{I}_{F_0}$

- Definition: a subset $S \subseteq \mathbb{Z}^+$ has *upper density*

$$\overline{\delta}(S) := \limsup_{x \to \infty} \frac{\#(S \cap [1, x])}{x}.$$

- Say that a family $\mathcal{F}$ of elliptic curves is **typically bounded in torsion** if for all $\epsilon > 0$ there exists $B := B(\epsilon) > 0$ so that the set

$$\{d \in \mathbb{Z}^+ : \max_{\substack{E_{/F} \in \mathcal{F}: \\ [F:\mathbb{Q}]=d}} \#E(F)[\mathrm{tors}] \geq B\} \subseteq \mathbb{Z}^+$$

  has upper density $\leq \epsilon$.

- So for any $\epsilon > 0$, there is a bound on torsion subgroups which works over **any** degree, as long as one ignores a subset of degrees of arbitrarily small upper density.

1. $\mathcal{E}_{\mathrm{CM}} := \{\text{CM elliptic curves}\} \Rightarrow \mathcal{E}_{\mathrm{CM}}$ is typically bounded in torsion (Bourdon, Clark and Pollack, 2017).
2. $\mathcal{E} := \{\text{all elliptic curves}\} \Rightarrow \mathcal{E}$ is *not* typically bounded in torsion (Clark, Milosevic and Pollack, 2018).

P1: Given integers $\ell, n_0 \in \mathbb{Z}^+$ with ℓ prime, there exists $n := n(\mathcal{F}, \ell, n_0) \in \mathbb{Z}^+$ such that for all $E_{/F} \in \mathcal{F}$, if $E(F)[\ell^n] \smallsetminus E(F)[\ell^{n-1}] \neq \emptyset$ then

$$\ell^{n_0} \mid [F : \mathbb{Q}].$$

P2: There exists $c := c(\mathcal{F}) \in \mathbb{Z}^+$ such that for all primes $\ell \in \mathbb{Z}^+$ and all $E_{/F} \in \mathcal{F}$, if $E(F)[\ell] \smallsetminus \{O\} \neq \emptyset$ then

$$\ell - 1 \mid c[F : \mathbb{Q}].$$

Theorem (Theorem 3.2, Clark, Milosevic and Pollack, 2018).

If $\mathcal{F}$ satisfies **P1** *and* **P2,** *then $\mathcal{F}$ is typically bounded in torsion.*

- Recall the family

$$\mathcal{I}_{F_0} := \{E_{/F} : E \text{ is } \overline{\mathbb{Q}}\text{-isogenous to some } E'_{/F_0}\}.$$

- "Elliptic curves $\overline{\mathbb{Q}}$-isogenous to at least one F_0-rational elliptic curve."

Theorem 2 (G., 2022).

For any number field F_0, the family $\mathcal{I}_{F_0}$ is typically bounded in torsion.

P1: strong uniform bounds on levels of ℓ-adic representations

Using a uniform bound on ℓ-adic levels of non-CM elliptic curves over a fixed degree (proven by Arai (2008)) and later strengthened by Clark and Pollack (2018)), we have shown the following.

Theorem 5 (G., 2022).

For all $d_0 \in \mathbb{Z}^+$ and prime ℓ, there exists $A := A(d_0, \ell) \in \mathbb{Z}^+$ such that for all non-CM $E_{/F}$ with $[F : \mathbb{Q}] = d_0$, one has for all $n \geq A$ and all cyclic subgroups $C \subseteq E$ of order ℓ^n that

$$\ell^{n-A} \mid [F(C) : F].$$

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This is enough to prove **P1** for $\mathcal{I}_{F_0}$.

P1: Given integers $\ell, n_0 \in \mathbb{Z}^+$ with ℓ prime, there exists $n := n(\mathcal{I}_{F_0}, \ell, n_0) \in \mathbb{Z}^+$ such that for all $E_{/F} \in \mathcal{I}_{F_0}$, if $E(F)[\ell^n]^* \neq \emptyset$ then

$$\ell^{n_0} \mid [F : \mathbb{Q}].$$

P2: isogeny characters

P2: There exists $c := c(\mathcal{F}) \in \mathbb{Z}^+$ such that for all primes $\ell \in \mathbb{Z}^+$ and all $E_{/F} \in \mathcal{F}$, if $E(F)[\ell]^* \neq \emptyset$ then

$$\ell - 1 \mid c[F : \mathbb{Q}].$$

We prove that $\mathcal{I}_{F_0}$ satisfies **P2** using **isogeny characters.**

- Given $E_{/F}$ and prime ℓ, a cyclic subgroup $C \subseteq E[\ell]$ is called F-**rational** if it is G_F-stable.
- The resulting 1-D representation is the **isogeny character** of C,

$$r\colon G_F \to (\mathbb{Z}/\ell\mathbb{Z})^\times.$$

Back to the **LV hypotheses:**

Theorem (Larson and Vaintrob, 2014).

Let F_0 be a number field. Then for all $\ell \gg_{F_0} 0$, if $E_{/F_0}$ is an elliptic curve with an F_0-rational ℓ-isogeny, then for its isogeny character $r\colon G_{F_0} \to (\mathbb{Z}/\ell\mathbb{Z})^\times$ we have one of the following:

1. *There exists a CM elliptic curve E' defined over F_0 such that its CM field $K := End(E') \otimes \mathbb{Q}$ is with $K \subseteq F_0$; and E' has an F_0-rational ℓ-isogeny, whose isogeny character $s\colon G_{F_0} \to (\mathbb{Z}/\ell\mathbb{Z})^\times$ satisfies*
$$r^{12} = s^{12}.$$
2. *GRH fails for $F_0(\sqrt{-\ell})$, and*
$$r^{12} = \chi_\ell^6$$
where χ_ℓ is the mod-ℓ cyclotomic character.

- Thus, if F_0 has no rationally defined CM and GRH is true, then there are no F_0-rational isogenies of large prime degree.

- This has previously been applied towards proving a subfamily of $\mathcal{I}_{F_0}$ is typically bounded in torsion, conditionally.
- Define the subfamily

$$\mathcal{E}_{F_0} := \{E_{/F} : j(E) \in F_0\}.$$

Theorem (Clark, Milosevic and Pollack, 2018).

Let F_0 be a number field. Assuming the **LV hypotheses** *for F_0, the family $\mathcal{E}_{F_0}$ is typically bounded in torsion.*

- To prove **P2** for $\mathcal{E}_{F_0}$, they use a classification of subgroups of $\mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$.

Theorem (Serre-Dickson Classification).

Let $\ell \geq 5$ be a prime. Then for a subgroup $G \subseteq \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$:

1. *If $\ell \mid \#G$, then up to conjugacy one of the following holds:*
   a. *G contains $\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$.*
   b. *G is contained in $B(\ell)$.*
2. *If $\ell \nmid \#G$, then up to conjugacy one of the following holds:*
   a. *G is contained in $N_s(\ell)$ or $N_{ns}(\ell)$.*
   b. *The image $\overline{G}$ of G in $\mathrm{PGL}_2(\ell)$ is isomorphic to one of the groups A_4, S_4 or A_5.*

- $B(\ell) :=$ Borel subgroup (upper triangular).
- $N_s(\ell)$ (resp. $N_{ns}(\ell)$) $:=$ normalizer of split (resp. non-split) Cartan subgroup.
- $\mathrm{PGL}_2(\ell) := \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})/(\mathbb{Z}/\ell\mathbb{Z})^\times$.

Theorem (Serre-Dickson Classification).

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1. *If $\ell \mid \#G$, then up to conjugacy one of the following holds:*
   a. *G contains $\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$.* **doable.**
   b. *G is contained in $B(\ell)$.* **doable?**
2. *If $\ell \nmid \#G$, then up to conjugacy one of the following holds:*
   a. *G is contained in $N_s(\ell)$ or $N_{ns}(\ell)$.* **doable.**
   b. *The image $\overline{G}$ of G in $\mathrm{PGL}_2(\ell)$ is isomorphic to one of the groups A_4, S_4 or A_5.* **doable.**

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Let $\ell \geq 5$ be a prime. Then for a subgroup $G \subseteq \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$:

1. *If $\ell \mid \#G$, then up to conjugacy one of the following holds:*
   a. *G contains $\mathrm{SL}_2(\mathbb{Z}/\ell\mathbb{Z})$.* **doable.**
   b. *G is contained in $B(\ell)$.* **exclude via LV hypotheses!**
2. *If $\ell \nmid \#G$, then up to conjugacy one of the following holds:*
   a. *G is contained in $N_s(\ell)$ or $N_{ns}(\ell)$.* **doable.**
   b. *The image $\overline{G}$ of G in $\mathrm{PGL}_2(\ell)$ is isomorphic to one of the groups A_4, S_4 or A_5.* **doable.**

- $B(\ell) :=$ Borel subgroup (upper triangular).
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- $\mathrm{PGL}_2(\ell) := \mathrm{GL}_2(\mathbb{Z}/\ell\mathbb{Z})/(\mathbb{Z}/\ell\mathbb{Z})^\times$.

We've shown that assuming the **LV hypotheses** is unnecessary:

Theorem 6 (G., 2022).

For any number field F_0, the family $\mathcal{E}_{F_0}$ is typically bounded in torsion.

Corollary 7 (G., 2022).

The family $\mathcal{I}_{F_0}$ satisfies **P2,** *and thus is typically bounded in torsion.*

Continuing Work

Recall

$$\mathcal{I}_{F_0} := \{E_{/F} : E \text{ is } \overline{\mathbb{Q}}\text{-isogenous to some } E' \text{ with } j(E') \in F_0\}.$$

Theorem 0 (G., 2022).

$\mathcal{I}_{F_0}$ *is* **typically bounded in torsion.**

- The study of $\mathcal{I}_{F_0}$ is originally motivated by a study of $\mathbb{Q}$**-curves.**

- Given an elliptic curve $E_{/F}$, for each automorphism $\sigma \in G_{F_0}$ one produces a new elliptic curve $E^{\sigma}{}_{/\sigma(F)}$ by applying σ to the equation for E.
- Let us define the family of F_0-**curves**

$$\mathcal{Q}_{F_0} := \{E_{/F} : \forall \sigma \in G_{F_0},\ E \text{ is } \overline{\mathbb{Q}}\text{-isogenous to } E^{\sigma}{}_{/\sigma(F)}\}.$$

- "Geometrically isogenous to Galois conjugates".
- When $F_0 = \mathbb{Q}$, this is the well-studied family of $\mathbb{Q}$-curves.

$$\mathcal{E}_{F_0} \subseteq \mathcal{I}_{F_0} \subseteq \mathcal{Q}_{F_0}$$

- If $F/\mathbb{Q}$ has odd degree, then $E_{/F}$ is a $\mathbb{Q}$-curve iff it is isogenous to a $\mathbb{Q}$-rational j-invariant (Cremona and Najman, 2021).

Conjecture.

The family $\mathcal{Q}_{\mathbb{Q}}$ is typically bounded in torsion.

- Follows from a boundedness conjecture on rational points on Atkin-Lehner quotients $X^*(N)$.
- Can try to show this for *central* $\mathbb{Q}$-curves, whose torsion subgroups are better understood; see e.g. (Sairaiji and Yamauchi, 2008).

Polynomial Bounds

Theorem -1 (G., 2022).

For each $\epsilon > 0$ and **fixed** *geometric isogeny class $\mathcal{E}$, there exists a constant $C_\epsilon := C_\epsilon(\mathcal{E})$ for which there are polynomial bounds of the form $C_\epsilon \cdot d^{1+\epsilon}$ on the exponent of torsion subgroups from $\mathcal{E}$.*

- Can we produce **uniform** polynomial bounds on "F_0-rational" geometric isogeny classes?

Project A

Conjecture.

For each number field F_0, there exists polynomial bounds on $\mathcal{I}_{F_0}$: for each $\epsilon > 0$ there are $B := B(F_0) > 0$ and $C_\epsilon := C_\epsilon(F_0) > 0$ such that for all $E_{/F} \in \mathcal{I}_{F_0}$ one has

$$\exp E(F)[\text{tors}] \leq C_\epsilon \cdot d^{B+\epsilon}$$

and

$$\#E(F)[\text{tors}] \leq C_\epsilon \cdot d^{B+1+\epsilon}.$$

- For $E_{/F} \in \mathcal{I}_{\mathbb{Q}}$ with $[F : \mathbb{Q}]$ odd, one has

$$\#E(F)[\text{tors}] \leq 720720\sqrt{35} \cdot [F : \mathbb{Q}]^{1/2}$$

(consequence of Bourdon and Najman, 2022).

Project B

Conjecture.

Fix a number field F_0 and an **abelian variety** A_{0/F_0}. *Then there exists polynomial bounds on the geometric isogeny class of A_{0/F_0}.*

- Would need analogous results on N-adic indices of isogenous abelian varieties, as well as an open image theorem for $\rho_{A_0,N}\colon G_{F_0} \to \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$.
  - The latter exists for PPAV's A_{0/F_0} with $\mathrm{End}(A_0) = \mathbb{Z}$ and $\dim(A_0) \in \{2, 6\} \cup \{\text{odd } n \in \mathbb{Z}^+\}$ (Serre, 1986).

Thank you!