Topology on Modular Curves

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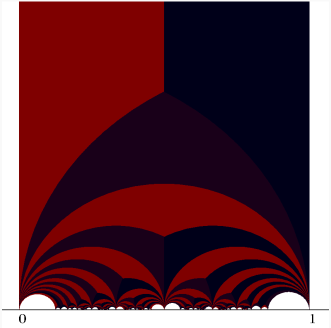

Group actions

- a group action G on a set X is a homomorphism into bijections:
  $$G \to \mathrm{Bij}(X).$$
  Namely, each $g \in G$ gives a bijection $g : X \to X$, and the maps $gh = g \circ h$.
- Group actions G on X partition the set into orbits
  $$\mathcal{O}(x) := \{g \cdot x : g \in G\}.$$
- So from an action G on X, one gets an equivalence relation: $x \sim y$ if $\exists g \in G$ with $x = g \cdot y$.
  - The equivalence classes are the orbits.

Quotient topologies

- Let X be a topological space. Suppose we have an equivalence relation $\sim$ on X.
- We have the projection map $\pi : X \to X/\sim$.
- We can endow $X/\sim$ with the *quotient topology*

$$T_{X/\sim} := \{A \subseteq X/\sim : \pi^{-1}(A) \text{ is open}\}.$$

  So open sets in $X/\sim$ are precisely those of the form $\pi(U)$ for open $U \subseteq X$.
- π is a continuous map.

The modular curve $Y(\Gamma)$ and its topology

- Fix a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$.
- Γ acts on $\mathbb{H}$ by linear fractional transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \tau := \frac{a\tau + b}{c\tau + d}.$$

- The **modular curve** $Y(\Gamma)$ is the quotient space of the orbits:

$$Y(\Gamma) := \mathbb{H}/\sim$$

  where $\tau \sim \gamma \cdot \tau$ for each $\tau \in \mathbb{H}$ and for all $\gamma \in \Gamma$.
- $Y(\Gamma)$ attains the quotient topology from $\mathbb{H}$ under this action.

- We would like to show that $Y(\Gamma)$ is a **Riemann surface**, i.e., a one-dimensional complex manifold.
- In particular, a Riemann surface is a connected Hausdorff space X (second-countable) with a chart, i.e., an open cover $\{U_i\}$ equipped with homeomorphisms $\psi_i : U_i \xrightarrow{\sim} V_i \subseteq \mathbb{C}$, so that on intersections one has that the *transition map*

$$\psi_j \circ \psi_i^{-1} : \psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$$

is differentiable.

- **Result 1:** $Y(\Gamma)$ is connected.
  - Proof: $\mathbb{H}$ is connected.
- **Result 2:** $Y(\Gamma)$ is Hausdorff.
- This follows from the action of Γ on $\mathbb{H}$ being **properly discontinuous:** that is, (almost) every point $\tau \in \mathbb{H}$ has a small enough neighborhood $U \subseteq \mathbb{H}$ so that for all $\gamma \in \Gamma$, if $\gamma \cdot U \cap U \neq \emptyset$ then $\gamma = \pm I$.
- Proper discontinuity follows from a more precise statement.

Proposition

For $\tau_1, \tau_2 \in \mathbb{H}$ there are neighborhoods U_1, U_2 of τ_1, τ_2 resp. in $\mathbb{H}$ so that for any $\gamma \in \Gamma(1)$,

$$\gamma \cdot U_1 \cap U_2 \neq \emptyset \Rightarrow \gamma \cdot \tau_1 = \tau_2.$$

Proof that $Y(\Gamma)$ is Hausdorff.

One can show that for $U_1, U_2 \subseteq \mathbb{H}$ one has

$$\pi(U_1) \cap \pi(U_2) = \emptyset \Leftrightarrow \Gamma \cdot U_1 \cap U_2 = \emptyset. \tag{1}$$

Pick $\tau_1, \tau_2 \in \mathbb{H}$. Let us choose resp. neighborhoods $U_1, U_2 \subseteq \mathbb{H}$ as per the proposition. Assuming $\pi(\tau_1) \neq \pi(\tau_2)$, this implies by the proposition that $\Gamma \cdot U_1 \cap U_2 = \emptyset$. Line (1) then shows us that $\pi(U_1), \pi(U_2)$ are disjoint neighborhoods of $\pi(\tau_1)$, $\pi(\tau_2)$. □

Corollary

Taking $\tau_1 = \tau_2 =: \tau$, *there is a neighborhood* U *of* τ *in* $\mathbb{H}$ *so that for any* $\gamma \in \Gamma(1)$

$$\gamma \cdot U \cap U \neq \emptyset \Rightarrow \gamma \cdot \tau = \tau.$$

- So the corollary says: each $\tau \in \mathbb{H}$ with trivial stabilizer $\pm I$ has a small enough neighborhood which contains no orbits $\gamma\tau \neq \tau$.

Corollary (of Corollary)

If $\tau \in \mathbb{H}$ has trivial stabilizer, then there is a neighborhood U of τ so that

$$\pi : U \to \pi(U)$$

is a homeomorphism.

- This corollary gives us our first idea for an atlas on $Y(\Gamma)$: we take as open sets the $U := \pi^{-1}(U_\tau)$ in that corollary, for each $\tau \in \mathbb{H}$.
- This is a good choice for a chart – except where is isn't, namely where $\mathrm{Stab}(\tau) \neq \pm I$.

- An example of a nontrivial stabilizer: negative inversion $S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ stabilizes i.
- One checks that around i, S acts at rotation by π.
- Therefore, any neighborhood U of i will contain two $\Gamma(1)$-equivalent points – thus, $\pi : U \to \pi(U)$ is not injective.

- Points $\tau \in \mathbb{H}$ with nontrivial stabilizer are called **elliptic points.**
- **Isotropy subgroup** refers to the stabilizer of τ in Γ.
- The existence of elliptic points forces us to modify our charts a bit. We will be changing our transition maps.
- In particular, we will transform our neighborhoods into a circular sector (an angular chunk of a circle) and then use a power map to fill in an open disk completely.

- We build our charts on $Y(\Gamma)$ as follows:
- For a number $\tau \in \mathbb{H}$, its *period* is the size of its stabilizer Γ_τ in $\Gamma \subseteq \Gamma(1) := \mathrm{SL}_2(\mathbb{Z})/\pm I$:

$$h_\tau := \#|\Gamma_\tau|.$$

  - Intuitively, the period is the number of sectors of a disk at the point that are identified under the stabilizer.
- The period h_τ is well-defined for $\pi(\tau) \in Y(\Gamma)$.
- A fact (Proposition 2.2.2 **[2]**) is that the isotropy subgroup Γ_τ is finite cyclic. Therefore, its period h_τ is finite.

Construction of charts

Our atlas will consist of open sets $U := U_\tau$ from the first proposition.
Let us fix $\tau \in \mathbb{H}$.

- The matrix $\delta_\tau := \begin{bmatrix} 1 & -\tau \\ 1 & -\overline{\tau} \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C})$ is so that: the stabilizer of 0 in $\delta_\tau \Gamma \delta_\tau^{-1}$ equals $\delta_\tau \Gamma_\tau \delta_\tau^{-1}$.
- Every matrix in $\delta_\tau \Gamma_\tau \delta_\tau^{-1}$ fixes 0 and ∞, so they are of the form $\gamma : z \mapsto Rz$ for $R \in \mathbb{C}$. In fact, since the order of such γ must divide h_τ, we have that $\gamma : z \mapsto k \cdot \frac{2\pi}{h_\tau} z$ for some $k \in \mathbb{Z}$.
- In particular, elements in the conjugate $\delta_\tau \Gamma_\tau \delta_\tau^{-1}$ are *rotations* by angular multiples of $\frac{2\pi}{h_\tau}$ about the origin.

- The correct charts of $Y(\Gamma)$ to take is as follows.
- For $\tau \in \mathbb{H}$, take $U := U_\tau$ as before. Then define the map

$$\psi : U \to \mathbb{C}, \ \psi(z) := \delta_\tau(z)^{h_\tau}.$$

- One checks that the images $\pi(U)$ and $\psi(U)$ are homeomorphic.
- One also checks that the transition maps are holomorphic.

Theorem

For any congruence subgroup Γ, $Y(\Gamma)$ is a Riemann surface.

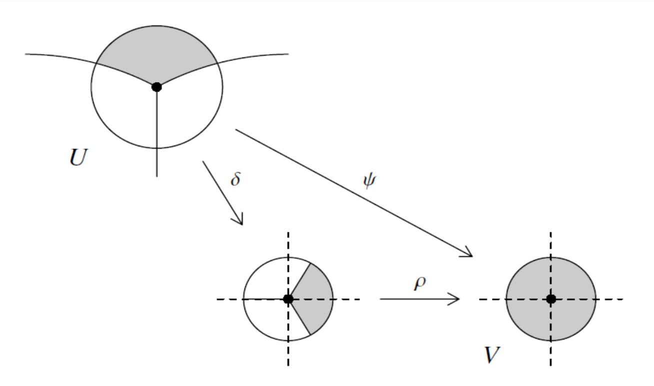

Figure: Depiction of the local maps $\rho(z) := z^{h_\tau}$, $\delta := \delta_\tau$ and $\psi := \rho \circ \delta$, **[2]**.

Fundamental domains

- For a congruence subgroup Γ, a *fundamental domain* $\mathcal{D}$ is a subspace of $\mathbb{H}$ which contains exactly one point from each orbit.
- For example, a fundamental domain for $\Gamma(1)$ is

$$\mathcal{D} = \left\{ \tau \in \mathbb{H} : \mathrm{Re}(z) \leq \frac{1}{2}, |\tau| \geq 1 \right\}$$

  (excluding some redundancies on the boundary).
- Morally, $\mathcal{D}$ is in bijection with the modular curve $Y(\Gamma)$.

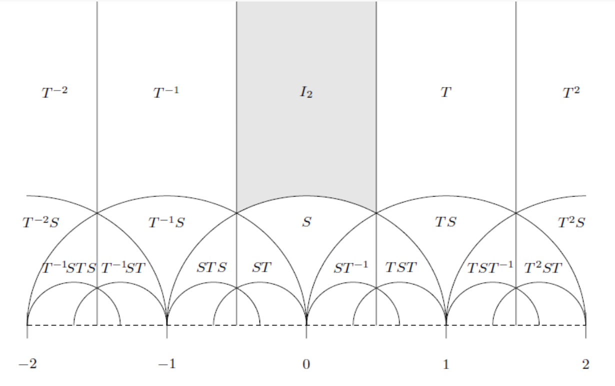

Figure: Translations $T : \tau \mapsto \tau + 1$ and negative inversions $S : \tau \mapsto -\frac{1}{\tau}$ of the fundamental domain for $\Gamma(1)$ (shaded), **[1]**. We note that S and T generate $\Gamma(1)$.

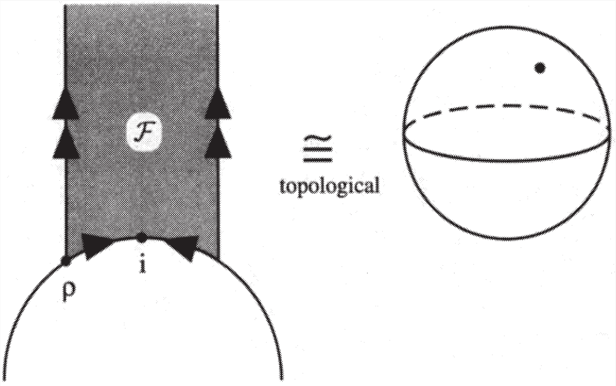

Figure: Using identifications on the boundary to see that the fundamental domain of $\Gamma(1)$ is a sphere, **[3]**.

- We see from the pictures that $Y(\Gamma(1))$ is not compact.
- To compactify $Y(\Gamma(1))$, and ultimately other modular curves $Y(\Gamma)$, we must add **cusps.**

Compact modular curves

- First, let us extend the upper half plane by adding a projective line:
$$\mathbb{H}^* := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}.$$
- So $z \in \mathbb{H} \cup \mathbb{Q}$ is identified with $\begin{bmatrix} z \\ 1 \end{bmatrix}$, while $\infty \rightsquigarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- We call the points on $\mathbb{P}^1(\mathbb{Q}) := \mathbb{Q} \cup \{\infty\}$ the **cusps** of $\mathbb{H}^*$.
- The action of $\Gamma(1)$ on $\mathbb{H}$ extends to $\mathbb{H}^*$: for $\gamma := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1)$,
$$\gamma \cdot \begin{bmatrix} x \\ y \end{bmatrix} := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
(matrix multiplication).

- Topologize $\mathbb{H}^*$ as follows: take as open basis the following:
  1. For $\tau \in \mathbb{H}$, all open disks in $\mathbb{H}$ centered at τ;
  2. for $\tau \in \mathbb{Q}$, all open disks centered at τ tangent to the real line, union $\{\tau\}$;
  3. for $\tau := \infty$, all upper half planes in $\mathbb{H}$ union $\{\infty\}$: namely, the

$$\mathbb{H}_r := \{\tau \in \mathbb{H} : \operatorname{im} \tau > r\} \cup \{\infty\}.$$

- As we will see, the quotient space $X(\Gamma) := \mathbb{H}^* / \sim$ under the action of Γ on $\mathbb{H}$ is a *compact* Riemann surface.
- $X(\Gamma)$ is the compactification of $Y(\Gamma)$.

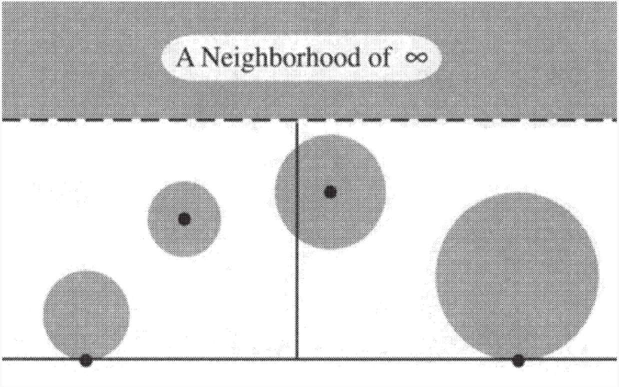

Figure: Examples of open sets on $\mathbb{H}^*$, **[3]**.

- We call the points of $X(\Gamma) \smallsetminus Y(\Gamma)$ the **cusps** of $X(\Gamma)$.
- One can show directly that $X(\Gamma(1))$ has only one cusp, and therefore $X(\Gamma)$ has *finitely* many cusps.

Topology on $X(\Gamma)$

- As mentioned earlier, $X(\Gamma)$ is a Riemann surface.
- Its charts are defined as follow; compare to the construction for elliptic points.
- For points $\tau \in \mathbb{H}$, we will keep the open sets U_τ as before, with their charts ψ.
- For each cusp $s \in \mathbb{P}^1(\mathbb{Q})$, fix a linear fractional transformation $\delta_s \in \Gamma(1)$ taking s to ∞. Define the *width* of s as

$$h_s := [\Gamma_\infty : (\delta_s \Gamma \delta_s^{-1})_\infty],$$

  where $\Gamma_\infty = \left\{ \pm \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} : m \in \mathbb{Z} \right\}$ is the isotropy subgroup of ∞ in $\Gamma(1)$.
- The width h_s of a cusp is finite and only depends on $\pi(\tau) \in X(\Gamma)$.

We are now ready to define charts on $X(\Gamma)$.

- For points $\tau \in \mathbb{H}$, take $U := U_\tau$ as before with their charts $\psi : \pi(U) \to U$.
- For a cusp $s \in X(\Gamma) \smallsetminus Y(\Gamma)$, let us set $U_s := \delta_s^{-1}(\mathbb{H}_2 \cup \{\infty\})$.
- Define

$$\psi : U_s \to \mathbb{C},\ \psi(z) := \exp\left(\frac{2\pi i}{h_s} \cdot \delta_s(z)\right).$$

Here is an explanation for the charts.

- δ_s was chosen to take a cusp s to the infinite cusp ∞.
- By definition, U_s is taken to the neighborhood $\mathbb{H}_2 \cup \{\infty\}$ of ∞ by δ_s.
- Then we take this neighborhood of ∞ to a sector of an open neighborhood of 0 by taking the exponential function;
- but in fact, with the appropriate scalar before applying e^z our sector is an open disk around 0 – we scale by $\frac{2\pi i}{h_s}$.
- One checks that this defines a chart on $X(\Gamma)$, whence $X(\Gamma)$ is a Riemann surface.

Finally, one may check that $X(\Gamma)$ is compact.

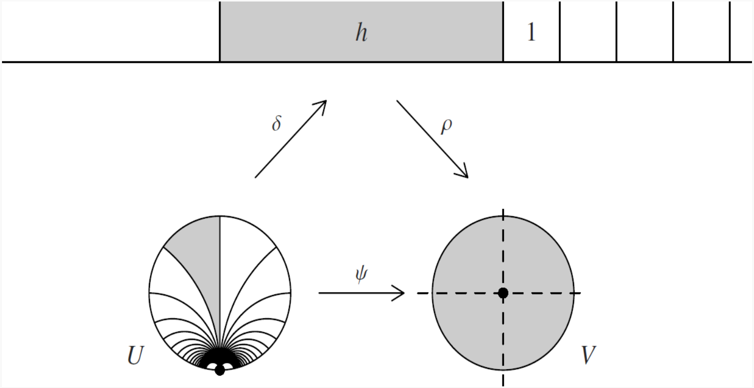

Figure: Depiction of the local maps $\rho(z) := \exp(2\pi iz/h_s)$, $\delta := \delta_s$ and $\psi := \rho \circ \delta$, **[2]**.

<table>
<tr><td colspan="2">$\pi : \mathcal{H}^* \longrightarrow X(\Gamma)$ is natural projection.
$U \subset \mathcal{H}^*$ is a neighborhood containing at most one elliptic point or cusp.
The local coordinate $\varphi : \pi(U) \xrightarrow{\sim} V$ satisfies $\varphi \circ \pi = \psi$
where $\psi : U \longrightarrow V$ is a composition $\psi = \rho \circ \delta$.</td></tr>
<tr><td>About $\tau_0 \in \mathcal{H}$:</td><td>About $s \in \mathbf{Q} \cup \{\infty\}$:</td></tr>
<tr><td>The straightening map is $z = \delta(\tau)$
where $\delta = \begin{bmatrix} 1 & -\tau_0 \\ 1 & -\overline{\tau}_0 \end{bmatrix}$, $\delta(\tau_0) = 0$.
$\delta(U)$ is a neighborhood of 0.</td><td>The straightening map is $z = \delta(\tau)$
where $\delta \in \mathrm{SL}_2(\mathbf{Z})$, $\delta(s) = \infty$.
$\delta(U)$ is a neighborhood of ∞.</td></tr>
<tr><td>The wrapping map is $q = \rho(z)$
where $\rho(z) = z^h$, $\rho(0) = 0$
with period $h = \lvert\{\pm I\}\Gamma_{\tau_0}/\{\pm I\}\rvert$.
$V = \rho(\delta(U))$ is a neighborhood of 0.</td><td>The wrapping map is $q = \rho(z)$
where $\rho(z) = e^{2\pi i z/h}$, $\rho(\infty) = 0$
with width $h = \lvert\mathrm{SL}_2(\mathbf{Z})_s/\{\pm I\}\Gamma_s\rvert$.
$V = \rho(\delta(U))$ is a neighborhood of 0.</td></tr>
</table>

Figure: A summary of local coordinates on $X(\Gamma)$, **[2]**.

References

[1] Keith Conrad, $SL_2(\mathbb{Z})$. https://kconrad.math.uconn.edu/blurbs/grouptheory/SL(2,Z).pdf.

[2] Fred Diamond and Jerry Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics 228, Springer-Verlag, 2005.

[3] Joseph Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics 151, Springer, 1994.