The Faltings Heights of CM Elliptic Curves and Special Gamma Values

Lindsay Cadwallader, Olivia Cannon, Tyler Genao

July 18, 2016

Elliptic curves

- An elliptic curve $E/\mathbb{Q}$ is given by an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

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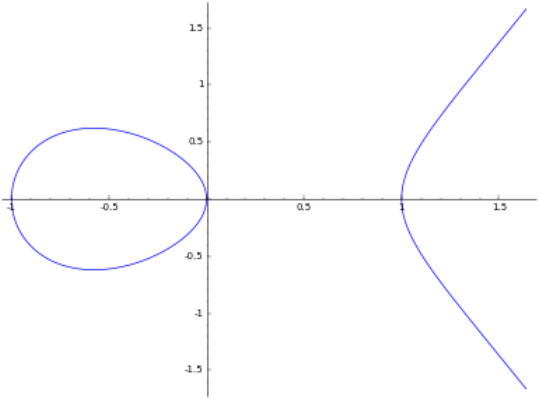
$$y^2 = x^3 + Ax + B$$

  for some $A, B \in \mathbb{Q}$.
- Define the discriminant of $E/\mathbb{Q}$ by

$$\Delta_E := -16(4A^3 + 27B^2) \neq 0.$$

Some examples



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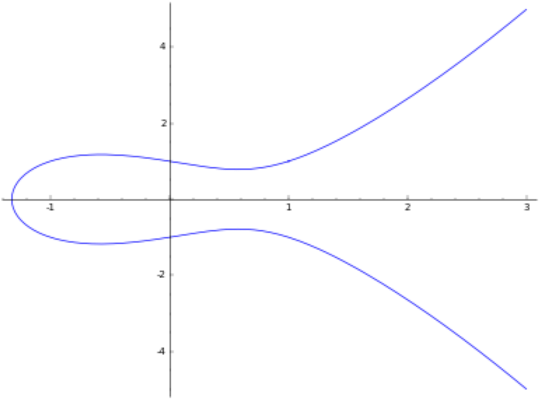

$$E/\mathbb{Q} : y^2 = x^3 - x + 1$$

Further definitions

- Define the j-invariant of $E/\mathbb{Q}$ by

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- Define the differential form ω_E by

$$\omega_E := \frac{dx}{2y}.$$

Lattices

- For two complex numbers ω_1 and ω_2, define the lattice generated by ω_1 and ω_2 by

$$L = L(\omega_1, \omega_2) := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{a\omega_1 + b\omega_2 : a, b \in \mathbb{Z}\}.$$

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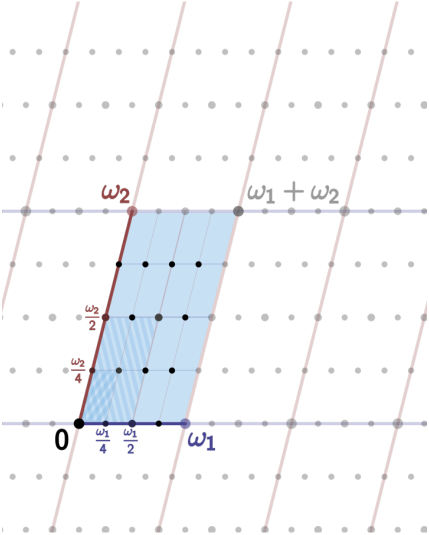
- Define the fundamental parallelogram associated to L by

$$P_L = P_{L(\omega_1,\omega_2)} := \{a\omega_1 + b\omega_2 : a, b \in [0, 1)\}.$$

A fundamental parallelogram



Uniformization

- One can prove that for any elliptic curve

$$E/\mathbb{C} : y^2 = x^3 + Ax + B,$$

there exists $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$ so that

$$E(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau).$$

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- For convenience, define

$$L_\tau := \mathbb{Z} + \mathbb{Z}\tau = [1, \tau].$$

The Faltings height of $E/\mathbb{Q}$

Definition

The Faltings height of $E/\mathbb{Q}$ is defined by

$$h_{\mathrm{Fal}}(E/\mathbb{Q}) := \frac{1}{12}\log|\Delta_E| - \frac{1}{2}\log\left(\frac{i}{2}\int_{E(\mathbb{C})}\omega_E \wedge \overline{\omega_E}\right).$$

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- If $E(\mathbb{C}) \cong \mathbb{C}/L_\tau$, then

$$\frac{i}{2}\int_{E(\mathbb{C})}\omega \wedge \bar{\omega} \ \sim_{\mathbb{Q}^\times} \ \mathrm{Area}(P_{L_\tau}).$$

Imaginary quadratic orders

- Let $d \in \mathbb{Z}$ be a negative squarefree integer and define the imaginary quadratic field

$$K := \mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} : a, b \in \mathbb{Q}\}.$$

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$$K := \mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} : a, b \in \mathbb{Q}\}.$$

- Define the discriminant of K by

$$D := \begin{cases} d & \text{if } d \equiv 1 \pmod 4, \\ 4d & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

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is called an imaginary quadratic order of conductor f in K.
- The order $\mathcal{O}_1$ is called the maximal order of K (the ring of integers of K).

CM elliptic curves

- For an elliptic curve $E/\mathbb{Q}$ and corresponding lattice $L = L_\tau$, define the endomorphism ring of $E/\mathbb{Q}$ by

$$\mathrm{End}_{\mathbb{C}}(E) := \{\alpha \in \mathbb{C} : \alpha L \subseteq L\}.$$

CM elliptic curves

Theorem

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- If $\mathrm{End}_{\mathbb{C}}(E)$ is isomorphic to $\mathcal{O}_f$, then $E/\mathbb{Q}$ is said to have complex multiplication (or CM) by $\mathcal{O}_f$.

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  at rational numbers.
- Our main result is an analogous formula for *any* order $\mathcal{O}_f$.

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- For an order $\mathcal{O}_f \subseteq K$, let $\Delta_f := f^2 D$ be the discriminant.
- Let $\chi_D(k)$ be the Kronecker symbol, which equals $-1, 0$, or 1 depending on the integer k.

Our Main Theorem

Theorem

If $E/\mathbb{Q}$ has CM by an order $\mathcal{O}_f \subset K$, then

$$h_{\mathrm{Fal}}(E/\mathbb{Q}) =$$

$$-\log\left(|\Delta_E|^{-1/12}\left(\frac{\pi}{\sqrt{|\Delta_f|}}\right)^{1/2}\prod_{k=1}^{|D|}\Gamma\left(\frac{k}{|D|}\right)^{\chi_D(k)\frac{\omega_D}{4h(D)}}\prod_{p|f}p^{e(p)/2}\right),$$

where

$$e(p) := -\frac{(1-p^{\mathrm{ord}_p(f)})(1-\chi_D(p))}{p^{\mathrm{ord}_p(f)-1}(1-p)(\chi_D(p)-p)}.$$

Examples of CM elliptic curves

$\mathcal{O}_f$	D	f	$E/\mathbb{Q}$
$\left[1, \frac{1+\sqrt{-3}}{2}\right]$	-3	1	$y^2 + y = x^3$
$[1, \sqrt{-3}]$	-3	2	$y^2 = x^3 - 15x + 22$
$\left[1, \frac{3+3\sqrt{-3}}{2}\right]$	-3	3	$y^2 + y = x^3 - 30x + 63$
$[1, i]$	-4	1	$y^2 = x^3 - x$
$[1, 2i]$	-4	2	$y^2 = x^3 - 11x - 14$
$\left[1, \frac{1+\sqrt{-7}}{2}\right]$	-7	1	$y^2 + xy = x^3 - x^2 - 2x - 1$
$[1, \sqrt{-7}]$	-7	2	$y^2 = x^3 - 595x - 5586$
$[1, \sqrt{-2}]$	-8	1	$y^2 = x^3 - x^2 - 3x - 1$
$\left[1, \frac{1+\sqrt{-11}}{2}\right]$	-11	1	$y^2 + y = x^3 - x^2 - 7x + 10$
$\left[1, \frac{1+\sqrt{-19}}{2}\right]$	-19	1	$y^2 + y = x^3 - 38x + 90$
$\left[1, \frac{1+\sqrt{-43}}{2}\right]$	-43	1	$y^2 + y = x^3 - 860x + 9707$
$\left[1, \frac{1+\sqrt{-67}}{2}\right]$	-67	1	$y^2 + y = x^3 - 7370x + 243528$
$\left[1, \frac{1+\sqrt{-163}}{2}\right]$	-163	1	$y^2 + y = x^3 - 2174420x + 1234136692$

An example of our Main Theorem

Consider the elliptic curve

$$E/\mathbb{Q} : y^2 = x^3 - 11x + 14,$$

which has CM by the order $\mathcal{O}_2 = \mathbb{Z} + \mathbb{Z}[2i] \subset \mathbb{Q}(i)$.

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- We have $f = 2$, $D = -4$, $\Delta_2 = -16$, $h(D) = 1$, $\omega_{-4} = 4$.
- The discriminant of $E/\mathbb{Q}$ is

$$\Delta_E = -16(4(-11)^3 + 27(14)^2) = 512 = 2^9.$$

From this, our Main Theorem gives

$$h_{\mathrm{Fal}}(E/\mathbb{Q}) = -\log\left(2^{-3/4}\left(\frac{\pi}{4}\right)^{1/2}\prod_{k=1}^{4}\Gamma\left(\frac{k}{4}\right)^{\chi_{-4}(k)} 2^{e(2)/2}\right).$$

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$$h_{\mathrm{Fal}}(E/\mathbb{Q}) = -\log\left(\frac{\pi^{1/2}}{2^{3/2}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)^{-1}\right).$$

Now, we can use the Gamma reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

to compute that

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Substituting this into the preceding expression gives

$$h_{\text{Fal}}(E/\mathbb{Q}) = -\log\left(\frac{1}{4\pi^{1/2}}\Gamma\left(\frac{1}{4}\right)^2\right).$$

Some additional definitions

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A fractional ideal $\mathfrak{a}$ is invertible if there exists a fractional ideal $\mathfrak{b}$ such that $\mathfrak{a}\mathfrak{b} = \mathcal{O}_f$. We call the group of invertible fractional ideals $I(\mathcal{O}_f)$

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The class number of $\mathcal{O}_f$, denoted $h(\mathcal{O}_f)$, is the number of such equivalence classes.

More background

Given a class $A \in \mathrm{Cl}(\mathcal{O}_f)$, we can choose an invertible ideal $\mathfrak{a}$ of $\mathcal{O}_f$ in A such that

$$\mathfrak{a} = \mathbb{Z}a + \mathbb{Z}\left(\frac{-b + \sqrt{\Delta_f}}{2}\right),$$

where $ax^2 + bx + c$ is a primitive positive definite quadratic form of discriminant Δ_f, with $a = N(\mathfrak{a})$.

More background

We have

$$\mathfrak{a}^{-1} = \mathbb{Z} + \mathbb{Z}\left(\frac{b+\sqrt{\Delta_f}}{2a}\right) = \mathbb{Z} + \mathbb{Z}z_{\mathfrak{a}^{-1}},$$

where

$$z_{\mathfrak{a}^{-1}} := \frac{b+\sqrt{\Delta_f}}{2a}$$

is the root in the upper-half plane of the quadratic form $ax^2 - bx + c$.

The Dedekind eta function

The Dedekind eta function is the weight $1/2$ modular form for $SL_2(\mathbb{Z})$ defined by

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q := e^{2\pi i z}$.

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For convenience, we define the $SL_2(\mathbb{Z})$-invariant function

$$F(z) := \sqrt{\mathrm{Im}(z)}|\eta(z)|^2.$$

A crucial Proposition

Proposition

If $E/\mathbb{Q}$ has CM by $\mathcal{O}_f$, then

$$h_{\mathrm{Fal}}(E/\mathbb{Q}) = -\log\left(2\pi|\Delta_E|^{-1/12}\prod_{[\mathfrak{a}]\in\mathrm{Cl}(\mathcal{O}_f)} F(z_{\mathfrak{a}^{-1}})^{1/h(\mathcal{O}_f)}\right).$$

A Chowla-Selberg formula for imaginary quadratic orders

Theorem

$$\prod_{[\mathfrak{a}]\in \mathrm{Cl}(\mathcal{O}_f)} F(z_{\mathfrak{a}^{-1}}) = \left(\frac{1}{4\pi\sqrt{|\Delta_f|}}\right)^{h(\mathcal{O}_f)/2} \prod_{k=1}^{|D|} \Gamma\left(\frac{k}{|D|}\right)^{\chi_D(k)\frac{\omega_D h(\mathcal{O}_f)}{4h(D)}} \prod_{p|f} p^{e(p)h(\mathcal{O}_f)/2},$$

where

$$e(p) := -\frac{(1-p^{\mathrm{ord}_p(f)})(1-\chi_D(p))}{p^{\mathrm{ord}_p(f)-1}(1-p)(\chi_D(p)-p)}.$$

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- We take an analytic approach by a direct study of the zeta function of a quadratic order.

The Zeta function

The zeta function of $\mathcal{O}_f$ is defined by

$$\zeta_{\mathcal{O}_f(s)} := \sum_{\substack{I \in I(\mathcal{O}_f) \\ 0 \neq I \subseteq \mathcal{O}_f}} \frac{1}{N(I)^s}, \quad \mathrm{Re}(s) > 1.$$

The fundamental identity

- For convenience, define the function

$$g_{\mathcal{O}_f}(s) := \frac{\#\mathcal{O}_f^\times}{2}\left(\frac{\sqrt{|\Delta_f|}}{2}\right)^{\frac{s+1}{2}}\frac{\zeta_{\mathcal{O}_f}\left(\frac{s+1}{2}\right)}{\zeta(s+1)}.$$

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- Recall that the $SL_2(\mathbb{Z})$ Eisenstein series is defined by

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- We show by an elaborate calculation that

$$g_{\mathcal{O}_f}(s) = \sum_{[\mathfrak{a}] \in \mathrm{Cl}(\mathcal{O}_f)} E\left(z_{\mathfrak{a}^{-1}}, \frac{s+1}{2} \right).$$

A renormalized Kronecker limit formula

Proposition

$$E\left(z, \frac{s+1}{2}\right) = 1 + \log(F(z))(s+1) + O((s+1)^2).$$

- The Taylor series expansion of $g_{\mathcal{O}_f}(s)$ at $s = -1$ is, formally,

$$g_{\mathcal{O}_f}(-1) + g'_{\mathcal{O}_f}(-1)(s+1) + O((s+1)^2).$$

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- It remains to calculate $g'_{\mathcal{O}_f}(-1)$.

The Taylor expansion of $g_{\mathcal{O}_f}(s)$

We have the factorization

$$\zeta_{\mathcal{O}_f}(s) = L_f(s)\zeta(s)L(\chi_D, s),$$

where

$$L_f(s) := \prod_{p|f} \frac{(1-p^{-s})(1-\chi_D(p)p^{-s}) - p^{\mathrm{ord}_p(f)(1-2s)-1}(1-p^{1-s})(\chi_D(p)-p^{1-s})}{1-p^{1-2s}}.$$

Therefore by an elaborate series of calculations we get that

$$g'_{\mathcal{O}_f} = \log\left(\left(\frac{1}{4\pi\sqrt{|\Delta|}}\right)^{\frac{h(\mathcal{O}_f)}{2}} \prod_{k=1}^{|D|} \Gamma\left(\frac{k}{|D|}\right)^{\chi_D(k)\frac{\omega_D \cdot h(\mathcal{O}_f)}{4h(D)}} \prod_{p|f} p^{e(p)\cdot\frac{h(\mathcal{O}_f)}{2}}\right)$$

Acknowledgements

We would like to thank Dr. Masri and Adrian for their help and guidance on this project, as well as the NSF for their generous support.