Primes dividing the ECHO sequence

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Outline

- Sequence and Elliptic Curves
- Numerical Approximations
- Galois Toolbox
- Calculating the Fraction
- Conclusion

Introduction

Question

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$\frac{2}{3}$ *(Lagarias).*

Lagarias and Hasse derived number fields with behaviors dependent entirely on whether p is a "good prime or not. They then calculated the density using the Chebotarev density theorem.

We do the same by analyzing Galois groups attached to elliptic curves.

Significance

Theorem (Jones and Rouse)

The Somos-4 sequence is defined by $a_0 = a_1 = a_2 = a_3 = 1$ *and further recursively defined by*

$$a_n a_{n-4} = a_{n-1} a_{n-3} + a_{n-2}^2.$$

The density of primes dividing a term of this sequence is $\frac{11}{21}$.

Proposition (Connection to Elliptic Curves (Jones and Rouse))

Let $E : y^2 + y = x^3 - x$ *and* $P = (0, 0)$ *be an elliptic curve and point. Then*

$$(2n-3)P = \left(\frac{a_n^2 - a_{n-1}a_{n+1}}{a_n^2}, \frac{a_{n-1}^2 a_{n+2} - 2a_{n-1}a_n a_{n+1}}{a_n^3} \right).$$

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- $$b_n = \begin{cases} \frac{b_{n-1}b_{n-3} - b_{n-2}^2}{b_{n-4}} & \text{if } n \not\equiv 2 \pmod 3, \\ \frac{b_{n-1}b_{n-3} - 3b_{n-2}^2}{b_{n-4}} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

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- The next few terms are

$$-7, \quad -17, \quad 2, \quad 101, \quad 247, \quad 571, \quad -1669,$$
$$-13766, \quad -43101, \quad -205897, \quad 1640929, \quad 8217293,$$
$$101727662, \quad 173114917, \quad -5439590147, \quad -70987557871, \ldots$$

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- Fact: $b_n \in \mathbb{Z}\ \forall n \geq 0$.

Elliptic Curves

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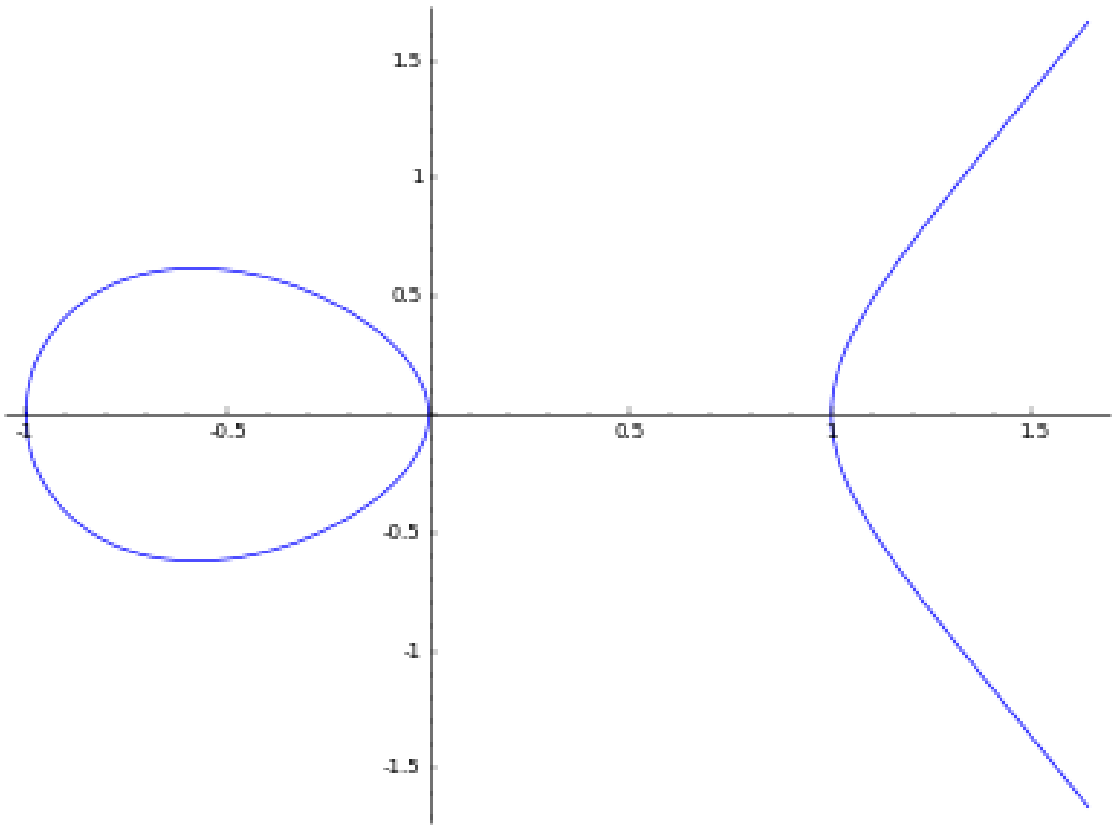
- Generally speaking, a normal elliptic curve E is a polynomial of the form $y^2 = x^3 + Ax + B$, where A, B are in a field $\mathbb{F}$.

Elliptic Curves (Examples)



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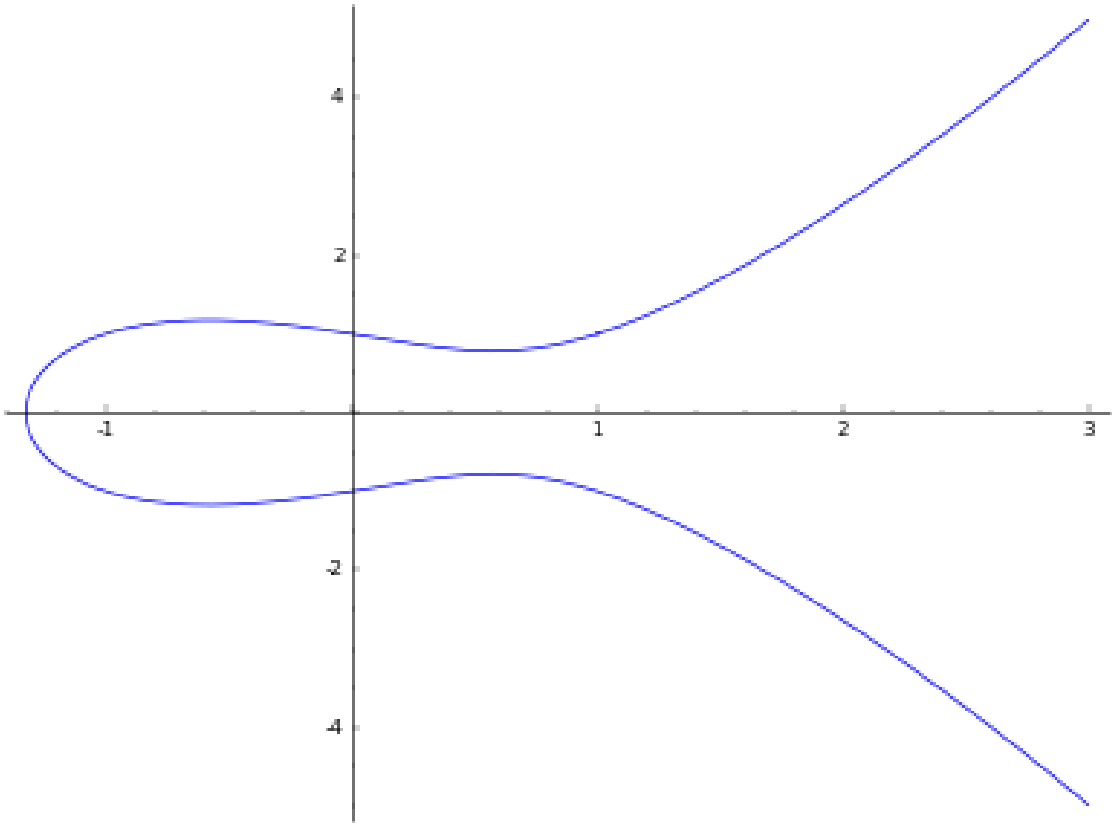
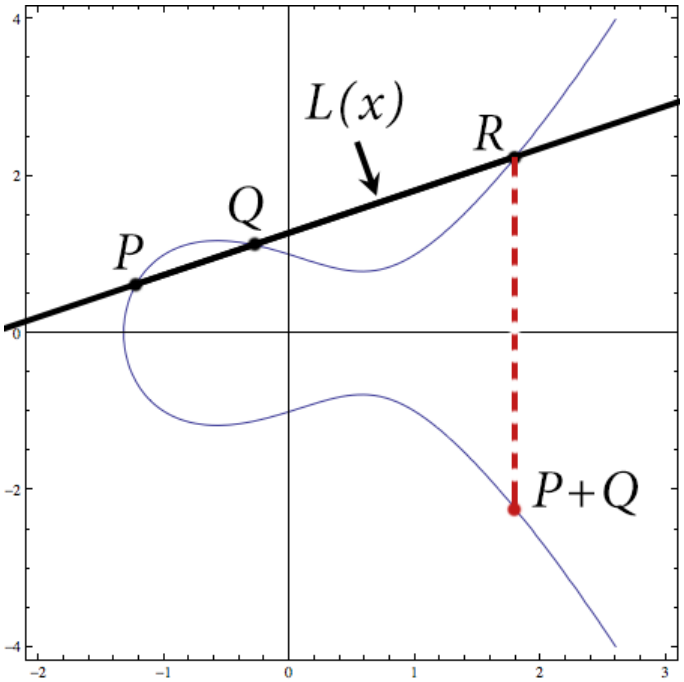


$$y^2 = x^3 - x + 1$$

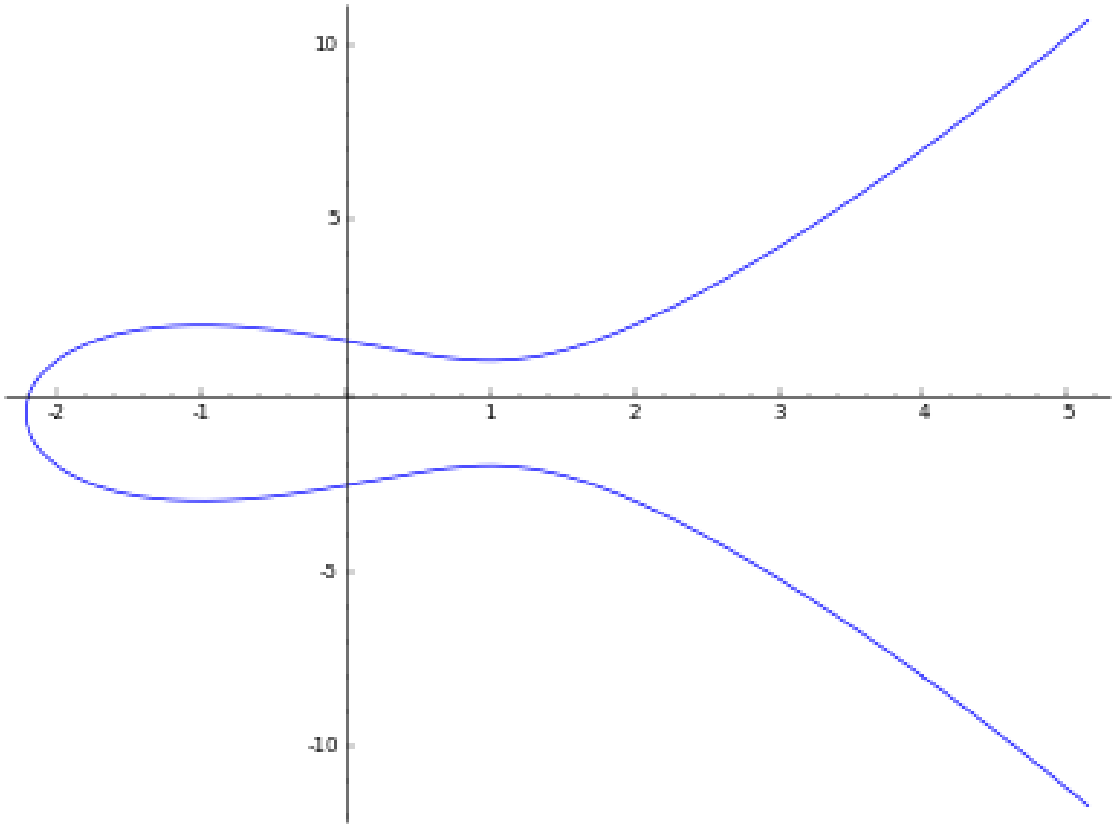
Elliptic Curves (Group Law)

- It turns out that in most cases, one can turn the curve into a group: if P and Q are two points on the curve, one can define the operation for P add Q: take the line intersecting both P and Q: it will intersect the curve at another point, say R. Then reflect that point over the y axis, and call this point $P + Q$.



The Elliptic Curve $y^2 + y = x^3 - 3x + 4$

- The elliptic curve we considered for this project was $E : y^2 + y = x^3 - 3x + 4$, and is pictured below:



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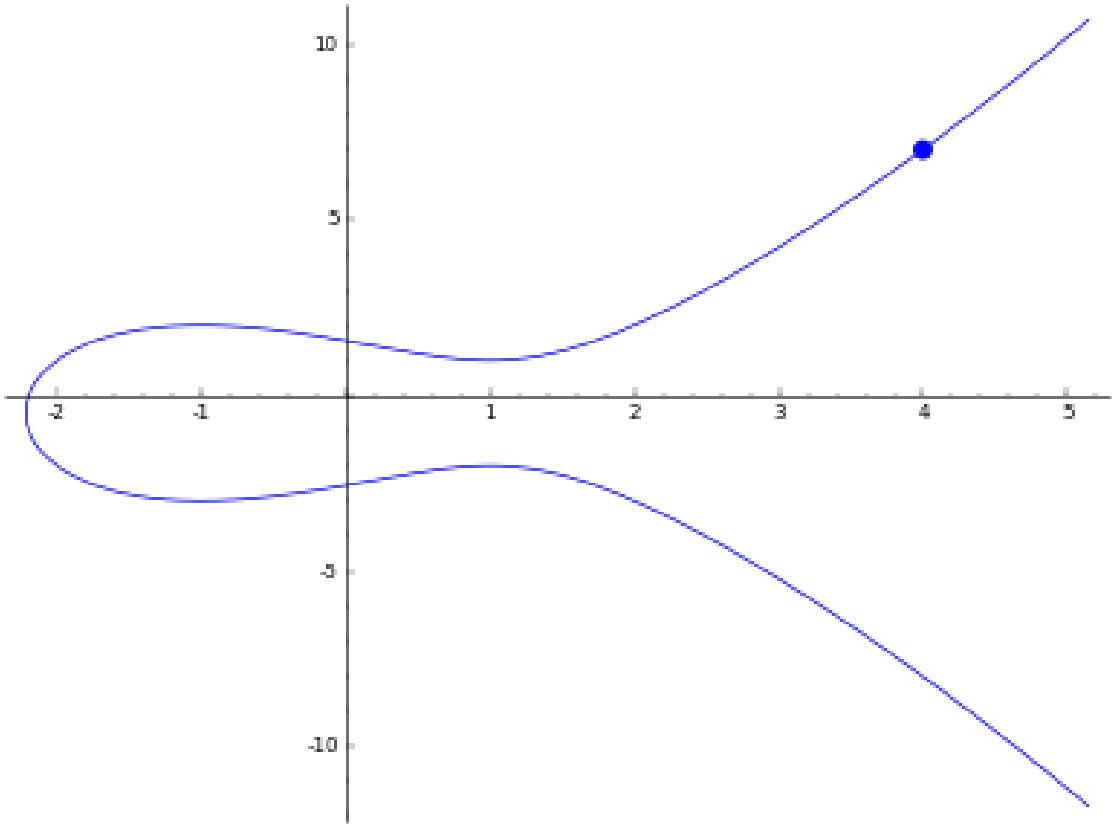

- A point on this curve is $P = (4, 7)$.

The Elliptic Curve $y^2 + y = x^3 - 3x + 4$

- It turns out that

$$(2n+3)P = \left(\frac{g(n)}{b_n^2}, \frac{f(n)}{b_n^3} \right)$$

where

$$g(n) = 2b_n^2 - b_{n-3}b_{n+3}$$

and

$$f(n) = \begin{cases} b_n^3 + b_{n-1}^2 b_{n+2} & \text{if } n \equiv 0 \pmod 3, \\ b_n^3 + 9b_{n-1}^2 b_{n+2} & \text{if } n \equiv 1 \pmod 3, \\ b_n^3 + 3b_{n-1}^2 b_{n+2} & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

and this point is in reduced form.

The Elliptic Curve $y^2 + y = x^3 - 3x + 4$

- Since
$$(2n+3)P = \left(\frac{g(n)}{b_n^2}, \frac{f(n)}{b_n^3}\right),$$
in projective coordinates we have
$$(2n+3)P = (b_n g(n) : f(n) : b_n^3).$$
Since the point is in reduced form, $\gcd(b_n, f(n)) = 1$.

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- So, the fraction of primes dividing the sequence is equivalent to the fraction of primes modulo which P has odd order.

Numerical Computation of the Fraction

Consider the fractions of $\pi(x)/\pi(x)$

x	$\pi(x)$	$\pi(x)$	$\frac{\pi(x)}{\pi(x)}$
10	3	4	0.75
10^2	13	25	0.52
10^3	91	168	0.541666667
10^4	636	1229	0.517493897
10^5	5118	9592	0.533569641
10^6	41856	78498	0.533211037
10^7	354158	664579	0.532905794
10^8	3069170	5761455	0.532707450
10^9	27092923	50847534	0.532826685
10^{10}	242426819	455052511	0.532744712
10^{11}	2193850226	4118054813	0.532739443

- $\frac{179}{336} \approx 0.532738095$

Understanding $Gal(\mathbb{Q}(E[N])/\mathbb{Q})$: Galois Representations

- Fact: $E[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$.
- We adjoin to $\mathbb{Q}$ the coordinates of the N-torsion points of E, and use the action of the Galois group on the torsion :

Definition

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- Some uses: determine the curve up to isogeny (Faltings), FLT

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- Given a curve E and a rational point P on E. Fix an N-division pt. of P, β.
- Beefed Up Galois Representation:

Definition

$$\omega_N : Gal(\mathbb{Q}([N]^{-1}P)/\mathbb{Q}) \longrightarrow AGL_2(\mathbb{Z}/N\mathbb{Z})$$

$$AGL_2(\mathbb{Z}/N\mathbb{Z}) := (\mathbb{Z}/N\mathbb{Z})^2 \rtimes GL_2(\mathbb{Z}/N\mathbb{Z})$$

$$\sigma \longmapsto (\sigma(\beta) - \beta, \rho_N(\sigma))$$

$AGL_2(\mathbb{Z}/2^k\mathbb{Z})$ and Primes dividing ECHO

- Note $a \in G$ has odd order iff for all $k \in \mathbb{Z}$ there is $\beta_k \in G$ such that $2^k \beta_k = a$
- Form the number field $K_k := \mathbb{Q}([2^k]^{-1}P)$ by adjoining the coordinates of all such β_k to $\mathbb{Q}$
- Want the primes p unramified such that $\sigma_p \in Gal(K_k/\mathbb{Q})$ fixes some $\beta_k : 2^k \beta_k = P$. If σ_p fixes β_k then $\beta_k \in E(\mathbb{F}_p)$ as desired.

Recall the homomorphism

$$\omega_k : Gal(K_k/\mathbb{Q}) \to AGL_2(\mathbb{Z}/2^k\mathbb{Z}) := (\mathbb{Z}/2^k\mathbb{Z})^2 \rtimes GL_2(\mathbb{Z}/2^k\mathbb{Z})$$

which is given by $\omega_k(\sigma_p) = (\vec{v}, M)$.

Definition

The action of $AGL_2(\mathbb{Z}/2^k\mathbb{Z})$ on $(\mathbb{Z}/2^k\mathbb{Z})^2$ is given by

$$(\vec{v}, M)(\vec{x}) = M\vec{x} + \vec{v}.$$

Theorem

There exists a fixed point β_k of σ_p iff $(M - I)\vec{x} = \vec{v}$ for some $\vec{x}$.

This is because σ_p has a fixed point iff $M\vec{x} + \vec{v} = \vec{x}$ for some $\vec{x}$, or equivalently $(M - I)\vec{x} = \vec{v}$, and any such fixed point for σ_p is a β_k.

Now we need only understand the image of ω_k in order to compute the fraction.

Kinetic Subgroups of $AGL_2(\mathbb{Z}/2^k\mathbb{Z})$

Definition (Kinetic)

Let a subgroup $G \subset AGL_2(\mathbb{Z}/2^k\mathbb{Z})$ for $k \geq 2$ be called kinetic if both the maps are surjective:

$$\text{pr} : G \twoheadrightarrow GL_2(\mathbb{Z}/2^k\mathbb{Z})$$
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Proposition

The image of ω_k must be kinetic.

Kinetic Subgroups of $AGL_2(\mathbb{Z}/2^k\mathbb{Z})$

Proposition

For all $k \geq 2$, there is one proper subgroup (up to conjugacy) $H_k \subset AGL_2(\mathbb{Z}/2^k\mathbb{Z})$ this is kinetic.

Proposition

For our curve, the image $\omega_k : \mathrm{Gal}(K_k/\mathbb{Q}) \to AGL_2(\mathbb{Z}/2^k\mathbb{Z})$ is H_k.

The fraction at the limit

Recall that by the Chebotarev density theorem

$$\lim_{x\to\infty} \frac{\left|\{p \text{ prime, unramified in } K_k : p \leq x, \left[\frac{K_k/\mathbb{Q}}{p}\right] \subseteq S\}\right|}{\pi(x)} = \frac{|S|}{|\operatorname{Gal}(K_k/\mathbb{Q})|}$$

where S is a union of conjugacy classes in $\operatorname{Gal}(K_k/\mathbb{Q})$. We need to evaluate this fraction for all $k \in \mathbb{Z}$.

The fraction at the limit

- We make two choices for S:
  1. Elements of $\mathrm{Gal}(K_k/\mathbb{Q})$ such that for all $(\vec{v}, M) \in H_k$, $\vec{v}$ is in the column space of $M - I$.
  2. same set except $(\vec{v}, M)$ such that $\det(M - I) \equiv 0 \mod (2^k)$
- We observe that the set of elements $(\vec{v}, M \in H_k)$ we desire for S sits between those two sets but at the limit as $k \to \infty$ the sizes of both sets over $|\mathrm{Gal}(K_k/\mathbb{Q})|$ are equal.

Theoretical Computation of the Fraction

Thus our final job was to compute the limit

$$\lim_{k\to\infty} \frac{|\{(\vec{v}, M) \in H_k | \vec{v} \in im(M - I)\}|}{|H_k|} = \left(lim_{k\to\infty} \frac{\pi(x)}{\pi(x)} \right).$$

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Our process: partition the numerator based on its reduction mod 4, because we understand the image at $k = 2$ very well.

The Answer

$$\frac{179}{336}$$

Future Work

Are there infinitely many curves $E/\mathbb{Q}$ for which there is a point $P \in E(\mathbb{Q})$ so that $\mathrm{im}(\omega_k) = H_k$?

- $E : y^2 + axy + by = x^3 + bx^2$
- $P = (0, 0)$
- Compute $f_{a,b}$, a two-parameter, degree 4 polynomial.

Summary of Results

Theorem

The density of primes dividing the ECHO sequence is the following:

$$\lim_{x\to\infty} \frac{\pi'(x)}{\pi(x)} = \frac{179}{336} \approx 0.532738095.$$

Theorem

Let E be an elliptic curve over $\mathbb{Q}$ and let P be a rational point on E. Suppose further that the image of the classical Galois representation is surjective, and that P has no rational 2-division points. Then there are only two possibilities for the image of the 2-adic arboreal representation up to conjugacy. As a consequence, the density of primes p for which the reduction of P modulo p has odd order is either $\frac{11}{21}$ or $\frac{179}{336}$.

Acknowledgements

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