Faltings Heights of CM Elliptic Curves

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Elliptic Curves

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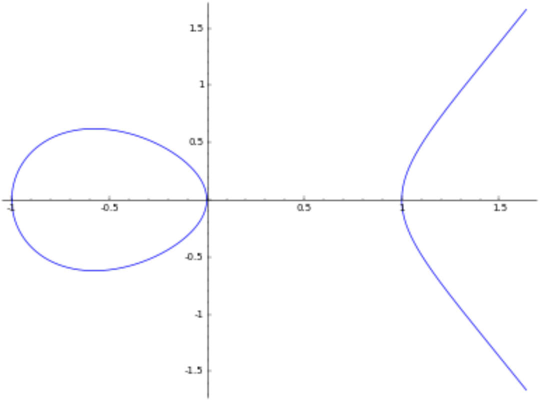
- Define the *discriminant* of $E/\mathbb{Q}$ to be

$$\Delta_E := -16(4A^3 + 27B^2).$$

Elliptic Curves (Examples)



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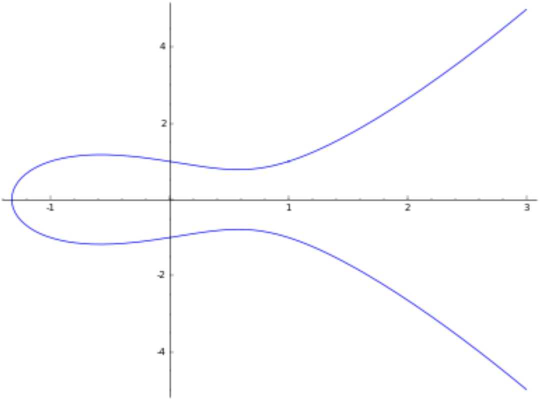
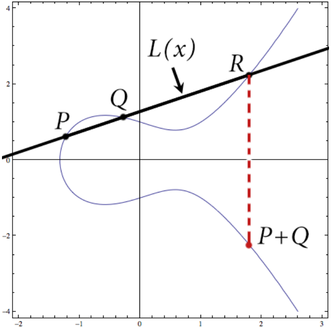


$$E/\mathbb{R} : y^2 = x^3 - x + 1$$

Elliptic Curves (Group Law)

- One can view the curve $E(\mathbb{R})$ as a group. If P and Q are two points on the curve, define the operation for P add Q like so: take the line intersecting both P and Q; it will intersect the curve at another point, say R. Then reflect that point over the y axis, and call this point $P + Q$.



Lattices

- For two complex numbers ω_1 and ω_2, define the *Lattice generated by ω_1 and ω_2* to be

$$L(\omega_1, \omega_2) := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{a\omega_1 + b\omega_2 : a, b \in \mathbb{Z}\}.$$

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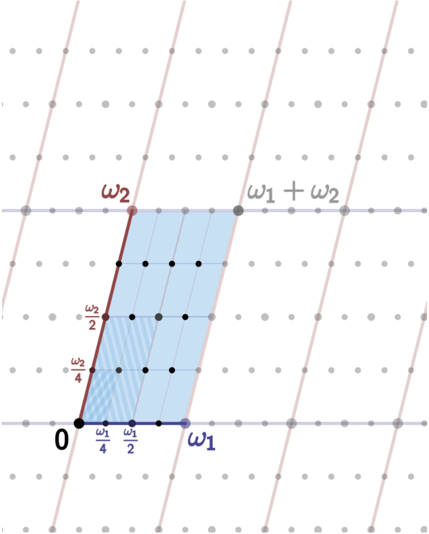
- The parallelogram $P_{L(\omega_1,\omega_2)}$ defined by ω_1 and ω_2 defines a *fundamental parallelogram* for $\mathbb{C}/L(\omega_1, \omega_2)$.

A Fundamental Parallelogram



Uniformization

- It turns out that for any elliptic curve

$$E/\mathbb{C} : y^2 = x^3 + Ax + B,$$

there exists $\tau \in \mathbb{C}$ with $\mathrm{Im}(\tau) > 0$ so that

$$E(\mathbb{C}) \cong \mathbb{C}/L(1, \tau).$$

The Faltings Height of an Elliptic Curve

Definition

$$h_{\mathrm{Fal}}(E/\mathbb{Q}) := \frac{1}{12}\log|\Delta_{E/\mathbb{Q}}| - \frac{1}{2}\log\left(\frac{i}{2}\int_{E(\mathbb{C})}\omega\wedge\bar{\omega}\right).$$

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Remark

$$\frac{i}{2}\int_{E(\mathbb{C})}\omega\wedge\bar{\omega} \;\sim_{\mathbb{Q}^\times}\; \mathrm{Area}(P_{L(1,\tau)}).$$

Imaginary Quadratic Orders

- For any negative squarefree $d \in \mathbb{Z}$, call the field

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an *imaginary quadratic number field*.

- Define the *discriminant* of K to be

$$D_K := \begin{cases} d & \text{if } d \equiv 1 \pmod 4, \\ 4d & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

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- For an integer $f > 0$, the ring

$$\mathcal{O}_f = [1, f\omega_K] := \{a + bf\omega_K : a, b \in \mathbb{Z}\}$$

  is called an *imaginary quadratic order of conductor* f in K.
- the order $\mathcal{O}_1$ of K is called the *maximal order* of K.

The Endomorphism Ring

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- Recall that for an elliptic curve $E/\mathbb{Q}$, there is a lattice $L = [1, \tau]$ such that $E(\mathbb{C}) \cong \mathbb{C}/L$.
- For an elliptic curve $E/\mathbb{Q}$ corresponding to a lattice L, we define the *endomorphism ring of $E/\mathbb{Q}$* to be

$$\mathrm{End}_{\mathbb{C}}(E) := \{\alpha \in \mathbb{C} : \alpha L \subseteq L\}.$$

CM Elliptic Curves

Theorem

For an elliptic curve $E/\mathbb{Q}$, $\mathrm{End}_{\mathbb{C}}(E)$ is isomorphic either to $\mathbb{Z}$ or to an order $\mathcal{O}_f$ in some K.

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- If $\mathrm{End}_{\mathbb{C}}(E)$ is isomorphic to $\mathcal{O}_f$, then $E/\mathbb{Q}$ is said to have *complex multiplication*, or *CM*.

Motivation

- For elliptic curves $E/\mathbb{Q}$ with CM by a maximal order, Deligne computed $h_{\mathrm{Fal}}(E/\mathbb{Q})$ in terms of Euler's Γ-function

$$\Gamma(s) := \int_0^\infty x^{s-1} e^{-x} dx$$

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  at rational numbers.
- Our main result is an analogous formula for any order $\mathcal{O}_f \subseteq K$.

Preliminaries to the Main Theorem

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- Let ω_{D_K} be 2, 4, or 6, depending on K.
- Let $h(K) < \infty$ denote the *class number* of K.
- Let $\chi_{D_K(k)}$ be -1, 0, or 1, depending on the integer k.

Theorem

The Faltings Height of an Elliptic Curve $E/\mathbb{Q}$ with CM

Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by an imaginary quadratic order $\mathcal{O}_f$ of K, with K having discriminant D_K. Then

$$h_{\mathrm{Fal}}(E/\mathbb{Q}) = -\log\left(|\Delta_{E/\mathbb{Q}}|^{-1/12}\left(\frac{\pi}{f\sqrt{|D_K|}}\right)^{1/2}\prod_{k=1}^{|D_K|}\Gamma\left(\frac{k}{|D_K|}\right)^{\chi_{D_K}(k)\,\frac{\omega_{D_K}}{4h(K)}}\prod_{p|f}p^{e(p)/2}\right),$$

where

$$e(p) = -\frac{(1-p^{\mathrm{ord}_p(f)})(1-\chi_D(p))}{p^{\mathrm{ord}_p(f)-1}(1-p)(\chi_D(p)-p)}.$$

Elliptic Curves with CM by Orders of Class Number One

$\mathcal{O}_f$	D	f	$E/\mathbb{Q}$
$\left[1, \frac{1+\sqrt{-3}}{2}\right]$	-3	1	$y^2 + y = x^3$
$[1, \sqrt{-3}]$	-3	2	$y^2 = x^3 - 15x + 22$
$\left[1, \frac{3+3\sqrt{-3}}{2}\right]$	-3	3	$y^2 + y = x^3 - 30x + 63$
$[1, i]$	-4	1	$y^2 = x^3 - x$
$[1, 2i]$	-4	2	$y^2 = x^3 - 11x - 14$
$\left[1, \frac{1+\sqrt{-7}}{2}\right]$	-7	1	$y^2 + xy = x^3 - x^2 - 2x - 1$
$[1, \sqrt{-7}]$	-7	2	$y^2 = x^3 - 595x - 5586$
$[1, \sqrt{-2}]$	-8	1	$y^2 = x^3 - x^2 - 3x - 1$
$\left[1, \frac{1+\sqrt{-11}}{2}\right]$	-11	1	$y^2 + y = x^3 - x^2 - 7x + 10$
$\left[1, \frac{1+\sqrt{-19}}{2}\right]$	-19	1	$y^2 + y = x^3 - 38x + 90$
$\left[1, \frac{1+\sqrt{-43}}{2}\right]$	-43	1	$y^2 + y = x^3 - 860x + 9707$
$\left[1, \frac{1+\sqrt{-67}}{2}\right]$	-67	1	$y^2 + y = x^3 - 7370x + 243528$
$\left[1, \frac{1+\sqrt{-163}}{2}\right]$	-163	1	$y^2 + y = x^3 - 2174420x + 1234136692$

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- $\Delta_E = -16(4(-11)^3 + 27(14)^2) = 512 = 2^9$.
- $\#\mathcal{O}_K^\times = 4$.

$$h_{\mathrm{Fal}}(E/\mathbb{Q}) = -\log\left(2^{-3/4}\left(\frac{\pi}{4}\right)^{1/2}\prod_{k=1}^{4}\Gamma\left(\frac{k}{4}\right)^{\chi_{-4}(k)}2^{e(2)/2}\right).$$

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So we have

$$h_{\mathrm{Fal}}(E/\mathbb{Q}) = -\log\left(\frac{\pi^{1/2}}{2^{3/2}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)^{-1}\right).$$

Furthermore, we can use the Gamma reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

to compute that

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Substituting this into the expression gives us that

$$h_{\mathrm{Fal}}(E/\mathbb{Q}) = -\log\left(\frac{1}{4\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)^2\right).$$

Thank you!